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 Quaternions and Small Lorentz Groups
 in Noncommutative Electrodynamics

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Non-linear electrodynamics arising in the frames of field theories in noncommutative space-time is examined on the base of quaternion formalism. The problem of form-invariance of the corresponding nonlinear constitutive relations governed by six noncommutativity parameters θ_{kl} or quaternion $\underline{K} = \underline{\theta} - i\underline{\epsilon}$ is explored in detail. Two Abelian 2-parametric small groups, $SO(2) \otimes SO(1.1)$ or T_2 , depending on invariant length $\underline{K}^2 \neq 0$ or $\underline{K}^2 = 0$ respectively, have been found. The way to interpret both small groups in physical terms consists in factorizing corresponding Lorentz transformations into Euclidean rotations and Lorentzian boosts.

In the context of general study of various dual symmetries in noncommutative field theory, it is demonstrated explicitly that the non-linear constitutive equations under consideration are not invariant under continuous dual rotations, instead only invariance under discrete dual transformation exists.

1 Introduction

It was James Clerk Maxwell who first used the quaternionic algebra by Sir William Rowan Hamilton to deal with electrodynamic equations. Since then, the areas of applying of quaternions steadily extend. Bi-quaternion algebra, that is the algebra by William Kingdon Clifford in 2-dimension complex space, turns to be especially useful in physical applications. The ground for so extensive employing the quaternions is that this formalism provides us with simple algebraic tools to deal with spinors of relativistic physics without any use of the cumbersome index technique [14].

The article aims to demonstrate the effectiveness of this algebraic technique to approach symmetries of Maxwell-like equations, equivalent in the sense by Seiberg – Witten [2] to the

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electrodynamic equations in a noncommutative space-time. As known [1], [2], [3], interest in field theory models in a noncommutative space-time has been grown notably after creating in [2] a general algorithm to relate usual Yang-Mills gauge models to their noncommutative counterparts. There appears a great deal of new physical problems to investigate, besides the question of the hypothetic coordinate non-commutativity has become of practically testable nature. Noticeable progress in describing symmetry of noncommutative spaces was achieved on the base of twisted Poincaré group [3], [4], [5], [6].

For instance, the mapping by Seiberg – Witten refers the noncommutative extension of electrodynamics to the usual microscopic Maxwell theory with special nonlinear constitutive relations. Examining all possible symmetries of these new constitutive relations seems to be a significant point in order to discern the effects of the space-time non-commutativity in observable electromagnetic non-linear effects.

The problem of form-invariance of the non-commutativity structural equations (see below) was considered in [4],[6]. Several simple noncommutative parameters were listed which allow for existence of some residual Lorentz symmetry – the later is recognized to have the structure $SO(2) \otimes SO(1,1)$. It was claimed in [4] that in the case of an arbitrary noncommutative matrix no residual Lorentz symmetry exists.

As known, the commutator for space-time coordinates \hat{x}^a in the Weyl-Moyal space is defined by an antisymmetric matrix θ^{ab} elements of which are real numbers:

$$[\hat{x}^\mu, \hat{x}^\nu] = i \theta^{\mu\nu} .$$

To any operator $f(\hat{x})$ there corresponds the Weyl symbol $f(x)$ defined on commutative Minkowski space with coordinates x^a . To the product of operators corresponds an operation $*$

$$(f * g)(x) = f(x) \exp \left(\frac{i}{2} \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu \right) g(x) \Big|_{x'=x} . \quad (1)$$

From (1) it follows

$$[x^\mu, x^\nu]_* = x^\mu * x^\nu - x^\nu * x^\mu = i \theta^{\mu\nu} . \quad (2)$$

Consistent use of the operation $*$ permits to define twisted analogue for Poincaré algebra [3-6] and provides us with possibility to construct representation of the Poincaré group. It was shown in [6] that at special particular choice of $\theta^{\mu\nu}$... the twisted Poincaré symmetry of noncommutative (quantum) field theory is reduced to the residual $O(1,1) \otimes SO(2)$ symmetry, but still carrying representations of the full Lorentz group. ... The meaning of the twisted Poincaré symmetry in NC QFT becomes transparent: it represents actually the invariance with respect to the stability group of $\theta_{\mu\nu}$, while the quantum fields carry representations of the full Lorentz group and the Hilbert space of states has the richness of particle representations of the commutative QFT".

In this context it is important to have known what is the full residual Lorentz symmetry for arbitrary matrix θ_{ab} ; because it was claimed in [4] that in the case of arbitrary noncommutative matrix no residual symmetry exists. There exist several different views [12] on the transforms of the matrix $\theta^{\mu\nu}$ under the Lorentz group, most radical attitude is to consider θ_{ab} as new

fundamental constants like the minimal length. In fact, our consideration does not depend on these different views.

In the paper, we use an quaternionic technique as a main tool for describing all Lorentz subgroups leaving invariant any 2-rank antisymmetric tensor. The treatment is given in the most general form irrespective of explicit vector constituents of the tensor. We place these quite conventional mathematical facts in the context of Maxwell equations in noncommutative space-time. As known according to of Seiberg – Witten [2], in the first order approximation in parameters $\theta^{\mu\nu}$ we get the ordinary Maxwell equations and special nonlinear constitutive relations. Just the symmetry properties of these nonlinear constitutive relations is the main subject of the paper. It permits us to make more clear and full the known results on residual Lorentz symmetry $SO(2) \otimes SO(1.1)$.

As mentioned, several particular examples of such small (or stability) subgroups were noticed in the literature, so our analysis extends and completes previous considerations. In a sense, the problem may be solved with the help of old and well elaborated technique in the theory of the Lorentz group. In the main parts, our examination of the problem is based on the use of bi-quaternion formalism [14]. Brief translation to the more traditional technique [7] developed for the Lorentz group and related to it will be given too, when instead of the antisymmetric tensor $\theta_{\mu\nu}$ we use a corresponding 3-vector under the complex orthogonal group $SO(3.C)$ which is equivalent to a symmetrical 2-rank spinor under $SL(2.C)$ ¹.

In the context of general study of various dual symmetries in noncommutative field theory [9], [10], [11], one other problem will be considered: it is demonstrated explicitly that the known nonlinear constitutive equations arising from noncommutative electrodynamics in the first order approximation are not invariant under continuous dual rotations, instead only invariance under discrete dual transformation exists, which contrasts with claim of the paper [9].

2 Residual Lorentz symmetry of the noncommutative Maxwell theory, quaternion treatment

It is known that extended Maxwell equations in noncommutative space-time, by means of Seiberg – Witten map in the first order approximation in parameters $\theta^{\mu\nu}$ provide us with the ordinary Maxwell equations and special nonlinear constitutive relations.

$$\text{div } \vec{B} = 0, \quad \text{rot } \vec{E} = -\frac{\partial \vec{B}}{\partial t}; \quad (3)$$

$$\text{div } \vec{D} = 0, \quad \text{rot } \vec{H} = \frac{\partial \vec{D}}{\partial t}, \quad (4)$$

and constitutive relations

$$\begin{aligned} \vec{D} &= \vec{E} + [(\vec{\epsilon}\vec{E}) - (\vec{\theta}\vec{B})] \vec{E} + [(\vec{\theta}\vec{E}) + (\vec{\epsilon}\vec{B})] \vec{B} + (\vec{E}\vec{B}) \vec{\theta} + \frac{1}{2}(\vec{E}^2 - \vec{B}^2) \vec{\epsilon}, \\ \vec{H} &= \vec{B} + [(\vec{\epsilon}\vec{E}) - (\vec{\theta}\vec{B})] \vec{B} - [(\vec{\theta}\vec{E}) + (\vec{\epsilon}\vec{B})] \vec{E} - (\vec{E}\vec{B}) \vec{\epsilon} + \frac{1}{2}(\vec{E}^2 - \vec{B}^2) \vec{\theta}, \end{aligned} \quad (5)$$

¹More detailed treatment of the problem in the frame of Rieman-Zilberstein-Majorana-Oppenheimer formalism and conventional spinor formalism will be published elsewhere.

and inverse relations (within the accuracy of first order terms)

$$\begin{aligned}\vec{E} &= \vec{D} + [\vec{\theta}\vec{H} - \vec{\epsilon}\vec{D}] \vec{D} - [\vec{\theta}\vec{D} + \vec{\epsilon}\vec{H}] \vec{H} - (\vec{D}\vec{H}) \vec{\theta} + \frac{1}{2} (\vec{H}^2 - \vec{D}^2) \vec{\epsilon}, \\ \vec{B} &= \vec{H} + [\vec{\theta}\vec{H} - \vec{\epsilon}\vec{D}] \vec{H} + [\vec{\theta}\vec{D} + \vec{\epsilon}\vec{H}] \vec{D} + (\vec{D}\vec{H}) \vec{\epsilon} + \frac{1}{2} (\vec{H}^2 - \vec{D}^2) \vec{\theta}.\end{aligned}\quad (6)$$

The conventional notation is used:

$$E^m = F^{m0}, \quad B^k = -\frac{1}{2}\epsilon^{klm}F_{lm}, \quad \epsilon^m = \theta^{m0}, \quad \theta^k = -\frac{1}{2}\epsilon^{klm}\theta_{lm};$$

$\vec{\epsilon}$ and $\vec{\theta}$ appear to be parameters of effective nonlinear media. In general, each of equations (3) and (4) exhibits a 20-dimensional Lie group symmetry [13]; in which manner the presence of the nonlinear constitutive equations (5) and (6) must constrict this symmetry. This is a main question. In solving the problem we will apply quaternion technique [14].

Any element in bi-quaternion algebra (algebra over complex numbers) can be presented as²

$$q = q_0 e_0 + q_a e_a = q_0 + \underline{q} = q_s + q_v,$$

where basic quaternions obey

$$e_0 e_a = e_a e_0, \quad e_0^2 = e_0, \quad e_a e_b = -\delta_{ab} e_0 + \epsilon_{abc} e_c,$$

so the product of two quaternions is given by

$$q p = (q_0 p_0 - \vec{q} \vec{p}) e_0 + (q_0 \vec{p} + p_0 \vec{q} + \vec{q} \times \vec{p}) \vec{e}.$$

Two special operations for bi-quaternions are defined: quaternion conjugation

$$\bar{q} = q_0 - e_a q_a = q_0 - \underline{q}, \quad q \bar{q} = \bar{q} q = q_0^2 + \vec{q}^2, \quad (\overline{qp}) = \bar{p} \bar{q}$$

and complex conjugation

$$q^* = q_0^* - q_a^* e_a = q_0^* - \underline{q}^*.$$

With the use of notation

$$\begin{aligned}\nabla &= -i\partial_t e_0 + \underline{e} \nabla = -i\partial_t + \underline{\nabla}, \\ \underline{f} &= \underline{B} - i\underline{E}, \quad \underline{h} = \underline{H} - i\underline{D}\end{aligned}\quad (7)$$

Maxwell equations in media can be presented in the form of the quaternion equation [14]

$$\nabla [(\underline{B} - i\underline{E}) + (\underline{H} - i\underline{D})] + \{\nabla [(\underline{B} - i\underline{E}) - (\underline{H} - i\underline{D})]\}^* = 0. \quad (8)$$

The Lorentz invariance of these equations is realized by the following transforms³

$$\begin{aligned}\nabla' &= L \nabla \bar{L}^*, \quad L \bar{L} = k_0^2 + \underline{k}^2 = k_0^2 + \vec{k}^2 = 1, \\ (\underline{B}' - i\underline{E}') &= L^* (\underline{B} - i\underline{E}) \bar{L}^*, \quad (\underline{H}' - i\underline{D}') = L^* (\underline{H} - i\underline{D}) \bar{L}^*.\end{aligned}\quad (9)$$

²Ordinary 3-vectors are referred as \vec{a} , whereas \underline{p} designates a vector part of a quaternion.

³Quaternion L corresponds to a spinor matrix $B = k_0 - i\vec{\sigma} \vec{k} \in SL(2, \mathbb{C})$ of the complex linear group $SL(2, \mathbb{C})$, spinor covering of the Lorentz group; restrictive relation $L \bar{L} = 1$ is equivalent to separation special linear group by imposing $\det B = 1$.

The constitutive relations in quaternionic form look

$$\begin{aligned} (\underline{B} - i\underline{E}) &= (\underline{H} - i\underline{D}) + [(\underline{H} + i\underline{D}) (\underline{\theta} + i\underline{\epsilon})]_S (\underline{H} - i\underline{D}) + \frac{1}{2}(\underline{H} + i\underline{D})_S^2 (\underline{\theta} - i\underline{\epsilon}) , \\ (\underline{H} - i\underline{D}) &= (\underline{B} - i\underline{E}) - [(\underline{B} + i\underline{E}) (\underline{\theta} + i\underline{\epsilon})]_S (\underline{B} - i\underline{E}) - \frac{1}{2}(\underline{B} + i\underline{E})_S^2 (\underline{\theta} - i\underline{\epsilon}) . \end{aligned} \quad (10)$$

Under the Lorentz group the quaternion $\underline{\Phi} = (\underline{\theta} - i\underline{\epsilon})$ transforms according to the law

$$\underline{\Phi}' = L^* \underline{\Phi} \bar{L}^* . \quad (11)$$

We have arrived at a key relationship determining small Lorentz group for the noncommutativity object:

$$L^* \underline{\Phi} \bar{L}^* = \underline{\Phi} , \quad \text{or} \quad L^* \underline{\Phi} = \underline{\Phi} L^* . \quad (12)$$

it describes all inertial observers to which effects of non-commutativity will be seen exactly the same. It is convenient to introduce a new variable $\Phi^* = \varphi$, then eq. (12) reads

$$(k_0 + \underline{k}) \underline{\varphi} = \underline{\varphi} (k_0 + \underline{k}) . \quad (13)$$

Evidently, that eq. (14) is satisfied if and only if quaternions \underline{q} and $\underline{\varphi}$ are proportional to each other:

$$L_\varphi = (k_0 + w \underline{\varphi}) , \quad \underline{\varphi} = (\underline{\theta} + i\underline{\epsilon}) . \quad (14)$$

To proceed further in description of the subgroup (14) we should distinguish between two cases: $\underline{\varphi}^2 \neq 0$ and $\underline{\varphi}^2 = 0$.

In the first case one can introduce new parametrization in term of a complex angle and unit quaternion:

$$\underline{\varphi}^2 \neq 0 , \quad k_0 = \cos \chi , \quad \underline{\varphi} = \sqrt{\underline{\varphi}^2} \frac{\underline{\varphi}}{\sqrt{\underline{\varphi}^2}} = \sin \chi \hat{\underline{\varphi}} , \quad \hat{\underline{\varphi}}^2 = +1 ; \quad (15)$$

so that the small Lorentz group with simple Abelian multiplication law is given by

$$L_\varphi = \cos \chi + \sin \chi \hat{\underline{\varphi}} , \quad \chi'' = \chi' + \chi . \quad (16)$$

Taking matrix realization for quaternion units, $e_0 = I, e_a = -i\sigma_a$, we get an explicit spinor form for L_φ

$$L_\varphi = \cos \chi - i \sin \chi \vec{\sigma} (\vec{n} + i\vec{m}) , \quad \vec{n} + i\vec{m} = \frac{(\vec{\theta} + i\vec{\epsilon})}{\sqrt{\vec{\theta}^2 - \vec{\epsilon}^2 + 2i\vec{\theta}\vec{\epsilon}}} . \quad (17)$$

Immediately, particular examples when physical interpretation for χ -parameter is evident can be pointed out: Euclidean rotations and Lorentz boosts along vector \vec{n} :

$$\begin{aligned} \vec{n} \neq 0 , \vec{m} = 0 , \chi = \alpha + i0 , \quad L_\varphi &= \cos \alpha - i \sin \alpha \vec{\sigma} \vec{n} ; \\ \vec{n} \neq 0 , \vec{m} = 0 , \chi = 0 + i\beta , \quad L_\varphi &= \text{ch } \beta + \text{sh } \beta \vec{\sigma} \vec{n} . \end{aligned} \quad (18)$$

It should be added, that the case of arbitrary nonisotropic $\theta^{\mu\nu}$ with the help of special Lorentz transformation can be translated to a more simple form properties:

$$\begin{aligned} \vec{n} + i\vec{m} , \quad \vec{n}^2 - \vec{m}^2 = 1 = \text{inv} , \quad \vec{n} \vec{m} = 0 = \text{inv} &\implies \\ \vec{n}' + i \, 0 , \quad \vec{n}'^2 = 1 = \text{inv} , \quad L'_\varphi = \cos \chi + i \sin \chi \, \vec{\sigma} \vec{n}' . \end{aligned}$$

Thus, for arbitrary $\theta^{\mu\nu}$ -tensor we arrive at the corresponding small Lorentz group $SO(2) \otimes SO(1,1)$, just this structure was previously described with special choice of non-commutativity matrix in [4], [6].

In the second (isotropic) case the normalization condition $L \bar{L} = 1$ gives

$$\underline{\varphi}^2 = 0 , \quad k_0^2 + w^2 \underline{\varphi}^2 = 1 , \quad k_0 = \pm 1 ;$$

therefore now the small Lorentz group (see 14)) is specified by relations

$$L = \pm(1 + w \underline{\varphi}) , \quad \underline{\varphi}^2 = 0 , \quad w'' = w' + w , \quad (19)$$

where w is any complex number. This is an Abelian group of displacements in complex plane, T_2 .

For readers preferring the Maxwell theory in vector notation let us give translation to this language. Electromagnetic vectors make up two complex 3-vector under complex orthogonal group $SO(3.C)$:

$$\vec{f} = \vec{B} - i \vec{E} , \quad \vec{h} = \vec{H} - i \vec{D} , \quad \vec{K} = \vec{\epsilon} - i \vec{\theta} . \quad (20)$$

Complex orthogonal group may be defined as $2 \rightarrow 1$ mapping from $SL(2.C)$ – their elements are given by

$$SO(3.C) , \quad O(k) = \begin{vmatrix} 1 - 2(k_2^2 + k_3^2) & -2k_0k_3 + 2k_1k_2 & +2k_0k_2 + 2k_1k_3 \\ +2k_0k_3 + 2k_1k_2 & 1 - 2(k_3^2 + k_1^2) & -2k_0k_1 + 2k_2k_3 \\ -2k_0k_2 + 2k_1k_3 & +2k_0k_1 + 2k_2k_3 & 1 - 2(k_1^2 + k_2^2) \end{vmatrix} \quad (21)$$

governs their behavior under Lorentz group in accordance with $O(k) \vec{f} = \vec{f}'$, $O(k) \vec{h} = \vec{h}'$. One may note straightforwardly an identity

$$O(k_0, \vec{k}) \lambda \vec{k} = \lambda \vec{k} \quad (22)$$

which is a base to explore the problem of small groups for complex 3-vectors.

3 On discrete dual symmetry

Let us turn to Maxwell equation in quaternion form and rewrite them as follows

$$\nabla(\underline{f} + \underline{h}) + [\nabla(\underline{f} - \underline{h})]^* = 0 , \quad (23)$$

where $\underline{f} = \underline{B} - i\underline{E}$, $\underline{h} = \underline{H} - i\underline{D}$. This equation is invariant under dual rotation:

$$\begin{aligned} \underline{f} + \underline{h} &= \underline{G} , \quad \underline{G}' = e^{i\chi} \underline{G} , \\ \underline{f} - \underline{h} &= \underline{R} , \quad \underline{R}' = e^{-i\chi} \underline{R} . \end{aligned} \quad (24)$$

We adhere to [8] and take dual rotation for $\underline{K} = \underline{\theta} - i\underline{\epsilon}$ in the form $\underline{K}' = e^{i\chi} \underline{K}$. In these variables, the constitutive relations read

$$\begin{aligned}\underline{f} &= \underline{h} + (\underline{h}^* \underline{K}^*)_s \underline{h} + \frac{1}{2} (\underline{h}^* \underline{h})_s \underline{K} , \\ \underline{h} &= \underline{f} - (\underline{f}^* \underline{K}^*)_s \underline{f} - \frac{1}{2} (\underline{f}^* \underline{f})_s \underline{K} .\end{aligned}\tag{25}$$

Summing and subtracting these two equations we get

$$\begin{aligned}0 &= -\frac{1}{2} (\underline{R}^* \underline{K}^*)_s \underline{G} + \frac{1}{2} (\underline{G}^* \underline{K}^*)_s \underline{R} - \frac{1}{2} (\underline{G}^* \underline{R}^*)_s \underline{K} - \frac{1}{2} (\underline{R}^* \underline{G}^*)_s \underline{K} , \\ 2\underline{R} &= \frac{1}{2} (\underline{R}^* \underline{K}^*)_s \underline{R} + \frac{1}{2} (\underline{G}^* \underline{K}^*)_s \underline{G} + \frac{1}{2} (\underline{G}^* \underline{G}^*)_s \underline{K} + \frac{1}{2} (\underline{R}^* \underline{R}^*)_s \underline{K} .\end{aligned}\tag{26}$$

Requiring invariance of these two relation with respect to dual rotation

$$\begin{aligned}\underline{G} &= e^{-i\chi} \underline{G}' , & \underline{G}^* &= e^{i\chi} \underline{G}'^* , \\ \underline{R} &= e^{i\chi} \underline{R}' , & \underline{R}^* &= e^{-i\chi} \underline{R}'^* , \\ \underline{K} &= e^{-i\chi} \underline{K}' , & \underline{K}^* &= e^{i\chi} \underline{K}'^* ,\end{aligned}$$

we arrive at two equations: $e^{i\chi} = e^{-3i\chi}$, $e^{-i\chi} = e^{+3i\chi}$ with simple solution: $e^{i\chi} = 1, -1, +i, -i$. Therefore, only discrete dual transformation leaves invariant the nonlinear constitutive equations, it corresponds to $e^{i\chi} = \pm i$. Thus, the dual symmetry's status in noncommutative electrodynamics differs with that in ordinary linear Maxwell theory in commutative space, this fact is to be interpreted in physical terms.

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References

- [1] M.R. Douglas, N.A. Nekrasov, *Noncommutative Field Theory*. Rev. Mod. Phys. **73**, 977-1029 (2001).
- [2] N. Seiberg, E. Witten, *String theory and noncommutative geometry*. J. High Energy Phys. **9909**, 032 (1999) [hep-th/9908142].
- [3] M. Chaichian, P.P. Kulish, K. Nishijima, A. Tureanu, *On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT*. Phys. Lett. **B604**, 98-102 (2004) [hep-th/0408069].
- [4] L. Álvarez-Gaumé, M.A. Vázquez-Mozo, *General properties of noncommutative field theories*, Nucl. Phys. **B668**, 293-325 (2003) [hep-th/0305093].
- [5] G. Fiore, J. Wess, *On full twisted Poincaré symmetry and QFT on Moyal-Weyl spaces*. Phys. Rev. **D75**, 105022 (2007).

- [6] M. Chaichian, K. Nishijima, T. Salminen, A. Tureanu, *Noncommutative Quantum Field Theory: A Confrontation of Symmetries*. [hep-th/0805.3500].
- [7] F.I. Fedorov, *The Lorentz group*, Moscow, 1979.
- [8] Y. Abe, R. Banerjee, I. Tsutsui, *Duality Symmetry and Plane Waves in Non-Commutative Electrodynamics*. Phys. Lett. **B573**, 248-254 (2003) [hep-th/0306272].
- [9] P. Aschieri, *On Duality Rotations in Light-Like Noncommutative Electromagnetism*. Mod. Phys. Lett. **A16**, 163-170 (2001) [hep-th/0103150v1].
- [10] O.J. Ganor, G. Rajesh and S. Sethi, *Duality and non-commutative gauge theory*, Phys. Rev. **D62**, 125008 (2000) [hep-th/0005046].
- [11] S. Rey, R. von Unge, *S-duality, noncritical open string and noncommutative gauge theory*, Phys. Lett. **B499**, 215 (2001) [hep-th/0007089].
- [12] G. Amelino-Camelia, F. Brisce, G. Gubitosi, A. Marciano, P. Martinetti, F. Mercati, *Twisted Hopf symmetries of canonical noncommutative spacetimes and the no-pure-boost principle*, [arXiv:0709.4600v1]. 0709.4600.
- [13] V.I. Fushchich, A.G. Nikitin, *Symmetry of Maxwell equations*, Kiev, 1983.
- [14] A.V. Berezin, Yu.A. Kurochkin, E.A. Tolkachev, *Quaternions in relativistic physics*, (in Russian), Minsk, 1989.